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Linear Algebra and its Applications 430 (2009) 418–422

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Hyperplanes of embeddable Grassmannians arise from projective embeddings: A short proof

Bart De Bruyn

*Ghent University, Department of Pure Mathematics and Computer Algebra, Krijgslaan 281 (S22),
B-9000 Gent, Belgium*

Received 26 October 2007; accepted 4 August 2008

Available online 9 September 2008

Submitted by R.A. Brualdi

Abstract

In this note, we give an alternative and considerably shorter proof of a result of Shult [E.E. Shult. Geometric hyperplanes of embeddable Grassmannians, *J. Algebra* 145 (1992) 55–82] stating that all hyperplanes of embeddable Grassmannians arise from projective embeddings.

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AMS classification: 51A45; 51E20

Keywords: Grassmannian; Hyperplane; Projective embedding

1. Introduction

Let $n \geq 1$ and let \mathbb{K} be a field. Let V denote an $(n + 1)$ -dimensional vector space over \mathbb{K} and let $\text{PG}(n, \mathbb{K}) = \text{PG}(V)$ denote the projective space associated with V . Let $k \in \{0, \dots, n - 1\}$. Then the following point-line geometry $A_{n,k+1}$ can be defined:

- The points of $A_{n,k+1}$ are the k -dimensional subspaces of $\text{PG}(n, \mathbb{K})$.
- The lines of $A_{n,k+1}$ are the sets $A(\pi_1, \pi_2)$ of k -dimensional subspaces of $\text{PG}(n, \mathbb{K})$ which contain a given $(k - 1)$ -dimensional subspace π_1 and are contained in a given $(k + 1)$ -dimensional subspace π_2 ($\pi_1 \subset \pi_2$).
- Incidence is containment.

E-mail address: bdb@cage.ugent.be

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doi:10.1016/j.laa.2008.08.003

The geometry $A_{n,k+1}$ is called the *Grassmannian of the k -dimensional subspaces* of $\text{PG}(n, \mathbb{K})$. We will denote the point-set of $A_{n,k+1}$ by \mathcal{P} . If x and y are two points of $A_{n,k+1}$, then $d(x, y) := k - \dim(x \cap y)$ is the distance between x and y in the collinearity graph of Γ . A *hyperplane* of $A_{n,k+1}$ is a proper subspace of $A_{n,k+1}$ which meets every line of $A_{n,k+1}$.

Now, let $\bigwedge^{k+1} V$ denote the $(k+1)$ th exterior power of V . For every k -dimensional subspace $\alpha = \langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k+1} \rangle$ of $\text{PG}(n, \mathbb{K})$, let $e(\alpha)$ denote the point $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \dots \wedge \bar{v}_{k+1} \rangle$ of $\text{PG}(\bigwedge^{k+1} V)$. Notice that the point $e(\alpha)$ is independent of the generating set $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k+1}\}$ of the subspace α . The map e defines a full projective embedding of $A_{n,k+1}$ into $\text{PG}(\bigwedge^{k+1} V)$ which is called the *Grassmann-embedding* of $A_{n,k+1}$. If π is a hyperplane of $\text{PG}(\bigwedge^{k+1} V)$, then $e^{-1}(\pi \cap e(\mathcal{P}))$ is a hyperplane of $A_{n,k+1}$. We say that the hyperplane $e^{-1}(\pi \cap e(\mathcal{P}))$ arises from the Grassmann-embedding of $A_{n,k+1}$.

The aim of this note is to give a short and elementary proof of the following result due to Shult [2].

Theorem 1.1. *All hyperplanes of $A_{n,k+1}$ arise from the Grassmann-embedding of $A_{n,k+1}$.*

Theorem 1.1 has an equivalent formulation in terms of alternating k -linear forms. This fact together with results of Ronan [1] and Wells [3] was exploited in [2] to prove Theorem 1.1. The alternative proof for Theorem 1.1 which we will now give is considerably shorter and only uses basic projective geometry.

2. Some useful results

We continue with the notation of Section 1.

Lemma 2.1. *Every hyperplane H of $A_{n,k+1}$ is a maximal subspace of $A_{n,k+1}$.*

Proof. Let X_1 and X_2 be two points of $A_{n,k+1}$ not contained in H . Recall that $d(X_1, X_2) = k - \dim(X_1 \cap X_2)$. We prove by induction on $d(X_1, X_2)$ that X_1 and X_2 are contained in a path which entirely consists of points of $\mathcal{P} \setminus H$. Obviously, this holds if $d(X_1, X_2) \leq 1$. So, suppose that $\delta := d(X_1, X_2) \geq 2$. For every $i \in \{1, 2\}$, let (y_i, α_i) be a non-incident point-hyperplane pair of X_i such that $X_1 \cap X_2 \subseteq \alpha_i$. Put $\beta_1 := \langle X_1, y_2 \rangle$ and $\beta_2 := \langle X_2, y_1 \rangle$. Then $A(\alpha_1, \beta_1)$ and $A(\alpha_2, \beta_2)$ are two lines of $A_{n,k+1}$. Moreover, for every point Z_1 of $A(\alpha_1, \beta_1)$, there exists a unique point $Z_2 \in A(\alpha_2, \beta_2)$ at distance $d(X_1, X_2) - 1$ from Z_1 , namely $Z_2 = \langle \alpha_2, z \rangle$, where z is the unique point in $Z_1 \cap y_1 y_2$. Since $X_i \notin H$ and $X_i \in A(\alpha_i, \beta_i)$, $|A(\alpha_i, \beta_i) \cap H| = 1$. So, it is possible to choose a $Z_1 \in A(\alpha_1, \beta_1)$ and a $Z_2 \in A(\alpha_2, \beta_2)$ such that $Z_1 \notin H$, $Z_2 \notin H$ and $d(Z_1, Z_2) = d(X_1, X_2) - 1$. By the induction hypothesis, Z_1 and Z_2 are connected by a path entirely consisting of points of $\mathcal{P} \setminus H$. Hence, also X_1 and X_2 are connected by such a path. \square

Suppose now that H_1 and H_2 are two distinct hyperplanes of $A_{n,k+1}$. Let Γ be the graph with vertex set $\mathcal{P} \setminus (H_1 \cup H_2)$, with two vertices x and y adjacent if and only if $d(x, y) = 1$ and the line xy meets $H_1 \cap H_2$. Let \mathcal{C} denote the set of all connected components of Γ and put $\mathcal{H} := \{H_1, H_2\} \cup \{C \cup (H_1 \cap H_2) \mid C \in \mathcal{C}\}$.

Lemma 2.2. *If H is a hyperplane of $A_{n,k+1}$ such that $H \cap H_1 = H_1 \cap H_2 = H \cap H_2$, then $H \in \mathcal{H}$.*

Proof. Since H_1 and H_2 are distinct maximal subspaces, $H_1 \cap H_2$ is not a maximal subspace. Since $H_1 \cap H_2 \subseteq H$ and H is a maximal subspace, there exists an $x^* \in H \setminus (H_1 \cap H_2)$. Clearly, $x^* \notin H_1 \cup H_2$. So, x^* is a vertex of Γ and there exists a unique element $W \in \mathcal{H}$ containing x^* . We will prove that $H = W$.

We first show that $W \subseteq H$. In view of the fact that $x^* \in W \cap H$, we need to show that if $x \in H \setminus (H_1 \cap H_2)$ and y is a vertex of Γ adjacent to x , then also $y \in H \setminus (H_1 \cap H_2)$. Now, since (i) $d(x, y) = 1$, (ii) xy meets $H_1 \cap H_2 = H \cap H_1$, and (iii) H is a subspace, it follows that $xy \subseteq H$. In particular, $y \in H$.

We next show that $H \subseteq W$. It suffices to prove the following (by induction on i): if $x, y \in H \setminus (H_1 \cap H_2)$ with $d(x, y) = i$ and $x \in W$, then also $y \in W$. The claim then immediately follows from the fact that $x^* \in H \cap W$. If $d(x, y) = 1$, then the line xy meets $H \cap H_1 = H_1 \cap H_2$. Hence, x and y are adjacent points of Γ and $y \in W$. So, we will suppose that $\delta = d(x, y) \geq 2$. We show that there exists a point u_x of x , a point u_y of y , a hyperplane α_x of x and a hyperplane α_y of y such that $x \cap y \subseteq \alpha_x$, $x \cap y \subseteq \alpha_y$, $u_x \notin \alpha_x$, $u_y \notin \alpha_y$, $\langle \alpha_x, u_y \rangle \in H$ and $\langle \alpha_y, u_x \rangle \in H$.

Let α_x be an arbitrary hyperplane of x through $x \cap y$ and let S denote the set of all k -dimensional subspaces of $\text{PG}(n, \mathbb{K})$ through α_x which intersect y in a subspace of dimension $k - \delta + 1$. Then S is a subspace of $A_{n,k+1}$ which carries the structure of a projective space isomorphic to $\text{PG}(\delta - 1, \mathbb{K})$. The set $S \cap H$ is equal to either S or a hyperplane of S (if we regard S as a projective space).

Suppose $\delta \geq 3$ or $S \cap H = S$. Let u_x be an arbitrary point of $x \setminus \alpha_x$. Let S' denote the set of all k -dimensional subspaces through u_x which intersect y in a hyperplane of y containing $x \cap y$. Then S' is a subspace of $A_{n,k+1}$ which carries the structure of a projective space $\text{PG}(\delta - 1, \mathbb{K})$. The set $S' \cap H$ is equal to either S' or a hyperplane of S' . If $\delta \geq 3$, then we see that there exist elements $\beta \in S \cap H$ and $\gamma \in S' \cap H$ such that $(\beta \cap y) \cap (\gamma \cap y) = x \cap y$. In this case, put $\alpha_y := \gamma \cap y$ and let u_y be an arbitrary point of $(\beta \cap y) \setminus (x \cap y)$. Suppose $\delta = 2$ and $S \cap H = S$. Let β be an arbitrary element of $S' \cap H$. Put $\beta \cap y = \alpha_y$ and let u_y be an arbitrary point of $y \setminus \alpha_y$. In both cases, one readily verifies that $(u_x, u_y, \alpha_x, \alpha_y)$ satisfies all required properties.

Suppose $\delta = 2$ and that $S \cap H$ is a singleton $\{\beta\}$. Let u_y be an arbitrary point of $(\beta \cap y) \setminus (x \cap y)$ and let α_y be an arbitrary hyperplane of y through $x \cap y$ not containing u_y . Let S' denote the set of all k -dimensional subspaces of $\text{PG}(n, \mathbb{K})$ through α_y which intersect x in a hyperplane of x . Then S' is a line. Since $\langle \alpha_y, \alpha_x \rangle \notin H$, there exists a unique element $\gamma \in S'$ belonging to H . Let u_x be an arbitrary point of $(\gamma \cap x) \setminus (x \cap y)$. Then $(u_x, u_y, \alpha_x, \alpha_y)$ satisfies all required properties.

Now, let u_x, u_y, α_x and α_y as above. Then $L_x := A(\alpha_x, \langle x, u_y \rangle)$ and $L_y := A(\alpha_y, \langle y, u_x \rangle)$ are lines of $A_{n,k+1}$. Since L_x contains the points x and $\langle \alpha_x, u_y \rangle$ of H , all the points of L_x are contained in H . Similarly, since L_y contains the points y and $\langle \alpha_y, u_x \rangle$ of H , all the points of L_y are contained in H . Clearly, every point z_1 of L_x has distance $d(x, y) - 1$ from a unique point z_2 of L_y , namely z_2 is the unique k -dimensional subspace containing α_y and the singleton $z_1 \cap u_x u_y$. Since $x \notin H_1$ and $y \notin H_1$, $|L_x \cap H_1| = 1 = |L_y \cap H_1|$. Hence, there exists a $z_1 \in L_x$ and a $z_2 \in L_y$ such that $z_1 \notin H_1$, $z_2 \notin H_1$ and $d(z_1, z_2) = d(x, y) - 1$. Now, applying the induction hypothesis three times, we find $z_1 \in W$, $z_2 \in W$ and $y \in W$. \square

Corollary 2.3. *If H_1 and H_2 arise from the Grassmann-embedding of $A_{n,k+1}$, then also every hyperplane H of $A_{n,k+1}$ satisfying $H \cap H_1 = H_1 \cap H_2 = H \cap H_2$ arises from the Grassmann-embedding of $A_{n,k+1}$.*

Proof. Take a point $x^* \in H \setminus (H_1 \cap H_2)$. Since H_i , $i \in \{1, 2\}$, is a maximal subspace, $\Sigma_i := \langle e(H_i) \rangle$ is a hyperplane of $\text{PG}(\bigwedge^{k+1} V)$. Moreover, $\Sigma_i \cap e(\mathcal{P}) = e(H_i)$. So, $\Sigma_1 \neq \Sigma_2$ and $e(x^*) \notin \Sigma_1 \cap \Sigma_2$ since $\Sigma_1 \cap \Sigma_2 \cap e(\mathcal{P}) = e(H_1) \cap e(H_2) = e(H_1 \cap H_2)$. Put $\Sigma := \langle e(x^*), \Sigma_1 \cap \Sigma_2 \rangle$ and $H' := e^{-1}(\Sigma \cap e(\mathcal{P}))$. Then $x^* \in H'$ and $H' \cap H_1 = H_1 \cap H_2 = H' \cap H_2$. By the proof of Lemma 2.2, $H = W = H'$, where W is the unique element of \mathcal{H} containing x^* . \square

3. Alternative proof of Theorem 1.1

We will prove Theorem 1.1 by induction on n . If $k \in \{0, n-1\}$, then $A_{n,k+1}$ is a projective space and the theorem trivially holds in this case. So, Theorem 1.1 holds if $n \leq 2$. In the sequel, we will suppose that $n \geq 3$ and $k \in \{1, \dots, n-2\}$.

Let (x, π) be a non-incident point-hyperplane pair of $\text{PG}(n, \mathbb{K})$. Let S_x , respectively S_π , be the subspace of $A_{n,k+1}$ consisting of all k -dimensional subspaces of $\text{PG}(n, \mathbb{K})$ which are incident with x , respectively π . The point-line geometry \tilde{S}_x (respectively \tilde{S}_π) induced on S_x (respectively S_π) is isomorphic to $A_{n-1,k}$ (respectively $A_{n-1,k+1}$). The Grassmann-embedding e of $A_{n,k+1}$ induces an embedding e_x of \tilde{S}_x into a subspace Σ_x of $\text{PG}(\bigwedge^{k+1} V)$ and an embedding e_π of \tilde{S}_π into a subspace Σ_π of $\text{PG}(\bigwedge^{k+1} V)$. Choosing a basis $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{n+1}\}$ in V such that $\langle \bar{e}_1 \rangle = x$ and $\langle \bar{e}_2, \dots, \bar{e}_{n+1} \rangle = \pi$, we see that: (i) Σ_x is the subspace of $\text{PG}(\bigwedge^{k+1} V)$ generated by all points of the form $\langle \bar{e}_1 \wedge \bar{f}_2 \wedge \dots \wedge \bar{f}_{k+1} \rangle$, where $\bar{f}_2, \dots, \bar{f}_{k+1}$ are vectors of $\langle \bar{e}_2, \dots, \bar{e}_{n+1} \rangle$; (ii) Σ_π is the subspace of $\text{PG}(\bigwedge^{k+1} V)$ generated by all points of the form $\langle \bar{f}_1 \wedge \bar{f}_2 \wedge \dots \wedge \bar{f}_{k+1} \rangle$, where $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_{k+1}$ are vectors of $\langle \bar{e}_2, \dots, \bar{e}_{n+1} \rangle$. Hence, Σ_x and Σ_π are complementary subspaces of $\text{PG}(\bigwedge^{k+1} V)$. It is also clear that e_x and e_π are isomorphic to the Grassmann-embeddings of respectively, $A_{n-1,k}$ and $A_{n-1,k+1}$.

Lemma 3.1. *Let y be a point of $A_{n,k+1}$ not contained in $S_x \cup S_\pi$. Then there exists a unique line L_y through y meeting S_x and S_π .*

Proof. Regarding y as a k -dimensional subspace of $\text{PG}(n, \mathbb{K})$, we have $x \notin y$ and $y \cap \pi$ is a $(k-1)$ -dimensional subspace of π . The line $L_y := A(y \cap \pi, \langle x, y \rangle)$ contains y , intersects S_x in the point $\langle x, y \cap \pi \rangle$ and S_π in the point $\langle x, y \rangle \cap \pi$. The uniqueness of L_y is also obvious. \square

Now, let H be an arbitrary hyperplane of $A_{n,k+1}$. Then $H \cap S_x$ is either S_x or a hyperplane of \tilde{S}_x . Similarly, $H \cap S_\pi$ is either S_π or a hyperplane of \tilde{S}_π . By Lemma 3.1, it is impossible that $H \cap S_x = S_x$ and $H \cap S_\pi = S_\pi$.

Suppose $H \cap S_\pi = S_\pi$. Then $H \cap S_x$ is a hyperplane of \tilde{S}_x . By the induction hypothesis, there exists a hyperplane β of Σ_x such that $H \cap S_x = e^{-1}(e(S_x) \cap \beta)$. Now, the hyperplane H is uniquely determined by $H \cap S_x$: a point $y \notin S_x \cup S_\pi$ is contained in H if and only if $L_y \cap S_x \subseteq H$. This implies that H is the hyperplane of $A_{n,k+1}$ arising from the hyperplane $\langle \beta, \Sigma_\pi \rangle$ of $\text{PG}(\bigwedge^{k+1} V)$. In a completely similar way, one shows that if $H \cap S_x = S_x$, then H arises from the Grassmann-embedding of $A_{n,k+1}$.

Suppose $H \cap S_x$ is a hyperplane of \tilde{S}_x and $H \cap S_\pi$ is a hyperplane of \tilde{S}_π . By the induction hypothesis, there exists a hyperplane β_1 of Σ_x and a hyperplane β_2 of Σ_π such that $H \cap S_x = e^{-1}(e(S_x) \cap \beta_1)$ and $H \cap S_\pi = e^{-1}(e(S_\pi) \cap \beta_2)$. Now, put $H_1 := e^{-1}(e(\mathcal{P}) \cap \langle \beta_1, \Sigma_\pi \rangle)$ and $H_2 := e^{-1}(e(\mathcal{P}) \cap \langle \beta_2, \Sigma_x \rangle)$. Then H_1 and H_2 are distinct hyperplanes of $A_{n,k+1}$. We show that

$$(H \cap L) \cap (H_1 \cap L) = (H_1 \cap L) \cap (H_2 \cap L) = (H \cap L) \cap (H_2 \cap L) \quad (1)$$

for every line L meeting S_x and S_π .

If $L \cap S_x \subseteq H$ and $L \cap S_\pi \subseteq H$, then $H \cap L = H_1 \cap L = H_2 \cap L = L$ and (1) holds. If $L \cap S_x \subseteq H$ and $L \cap S_\pi \cap H = \emptyset$, then $H \cap L = L \cap S_x$, $H_1 \cap L = L$, $H_2 \cap L = L \cap S_x$ and (1) holds again. A similar reasoning applies to the case $L \cap S_\pi \subseteq H$ and $L \cap S_x \cap H = \emptyset$. Finally, suppose $L \cap S_x \cap H = L \cap S_\pi \cap H = \emptyset$. Then $H_1 \cap L = L \cap S_\pi$, $H_2 \cap L = L \cap S_x$ and $H \cap L$ is a singleton different from $H_1 \cap L$ and $H_2 \cap L$. So, (1) holds again.

By Lemma 3.1 and (1), $H_1 \cap H = H_1 \cap H_2 = H_2 \cap H$. By Corollary 2.3, H arises from the Grassmann-embedding of $A_{n,k+1}$.

Acknowledgement

The author is a Postdoctoral Fellow of the Research Foundation – Flanders (Belgium).

References

- [1] M.A. Ronan, Embeddings and hyperplanes of discrete geometries, *European J. Combin.* 8 (1987) 179–185.
- [2] E.E. Shult, Geometric hyperplanes of embeddable Grassmannians, *J. Algebra* 145 (1992) 55–82.
- [3] A.L. Wells, Universal projective embeddings of the Grassmannian, half spinor, and dual orthogonal geometries, *Quart. J. Math. Oxford Ser. 34* (1983) 375–386.